ERRORS IN THE MEASUREMENT OF THE STEADY-STATE FLOW OF HEAT AT THE SURFACE OF A BODY

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We have undertaken an analysis of the distortions in the flow of heat in an object, said distortions attributable to the presence of a heat-metering thermometer, and we have derived formulas to estimate this measurement error.

It is impossible to measure the real value of the flow of heat in an object by means of a gradient-type heat-metering thermometer [1] mounted on the surface of an object because of the perturbations that this device introduces into the earlier existing distribution of temperatures. We know of a formulation of the problem of introducing correction factors into the readings of a heat-metering thermometer located at the surface of a body; however, the solution of this problem has not yet been brought to a form convenient for practical calculations [2].

Let us examine the case in which we measure the stationary flow of heat through the surface (half space) of a massive body with thermal conductivity λ by means of a gradient disk thermometer of radius R, thickness h, and thermal conductivity λ_t, mounted on the object in the manner shown in Fig. 1. It is assumed that the flow of heat through the object is formed under the action of an external radiant flux of density q, with the exchange of heat with the ambient medium, the latter exhibiting a temperature t. The intensity of the convection-radiation heat exchange with the medium is characterized by the heat-transfer coefficient α_0, while the absorption of the flow q is characterized by the absorption coefficient A_0 of the body.

The undistorted temperature field in the object (without the installation of the thermometer) is one-dimensional and near the surface of the object is determined by the relationship

$$t_0(z) = t_0 - \frac{q_0}{\lambda} z, \quad (1)$$

where the density of the heat flux entering the object (its true value) is given by

$$q_0 = -\lambda b = -\lambda \frac{dt_0}{dz}\Big|_{z=0} = \alpha_0 (t - t_0) + A_0 q.$$

Because of the differences between the thermophysical properties of the thermometer and the properties of the object, the mounting of the thermometer on the object changes the coefficients α_0 and A_0 by α_t and A_t and leads to the appearance of a new and distorted temperature field t(r, z) in the object. The density of the heat flow is given by

$$q_t(r) = \alpha_t [t - t_t(r)] + A_t q = \frac{\lambda_t}{h} \Delta t_t(r),$$

directed at the external surface of the thermometer, and creates the following temperature drop across the thickness of the thermometer:

$$\Delta t_t(r) = t_t(r) - t(r, z)|_{z=0}, \quad 0 \leqslant r \leqslant R.$$

The measured value of the heat flow q_e (the readings of the thermometer) will depend in the general case on the size of the area occupied by its sensitive element, the temperature values t(r, 0) within the object beneath the surface of the thermometer, as well as by the

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magnitudes of the radial flows of heat which arise within the thermometer itself. Taking the effect of the above-enumerated factors into full account requires analysis of the temperature field within the system formed by the thermometer and the object, and this must be compared with the undistorted field within the object.

If we neglect the radial flows in the thermometer (this results in elevated measurement errors), the problem of accounting for the distorted effect of the thermometer leads to the solution of the Laplace equation

$$\frac{\partial^2\vartheta}{\partial z^2}+\frac{\partial^2\vartheta}{\partial r^2}+\frac{1}{r}\frac{\partial\vartheta}{\partial r}=0 \qquad (2)$$

for the temperature perturbation $\vartheta(r, z) = t(r, z) - t_0(z)$ in the region $0 \le r < \infty$, $0 \le z < \infty$ under the boundary conditions of perturbation attenuation at a distance from the thermometer

$$\left.\frac{\partial\vartheta}{\partial z}\right|_{z\to\infty}=\left.\frac{\partial\vartheta}{\partial r}\right|_{r\to\infty}=0 \qquad (3)$$

and the boundary conditions

$$\left[\frac{\partial\vartheta}{\partial z}-\frac{\alpha_0}{\lambda}\vartheta-\frac{{}_\Delta\alpha}{\lambda}\vartheta\sigma(r-R)\right]\Bigg|_{z=0,\ 0\le r<\infty}={}_\Delta b\sigma(r-R), \qquad (4)$$

in which

$${}_\Delta\alpha=\frac{1}{1+\zeta_t}\alpha_t-\alpha_0,\quad \zeta_t=2\zeta_0\frac{1}{k}\frac{\lambda}{\lambda_t}\frac{\alpha_t}{\alpha_0},$$

$$\zeta_0=\frac{\alpha_0 R}{\lambda},\quad k=\frac{2R}{h},\quad {}_\Delta b=b_t-b,$$

$$b_t=-\frac{1}{\lambda(1+\zeta_t)}[\alpha_t(t-t_0)+A_t q],\quad b=-\frac{1}{\lambda}[\alpha_0(t-t_0)+A_0 q],$$

with the symbol $\sigma(r - R)$ denoting the unit function

$$\sigma(r-R)=\begin{cases}1, & 0\le r<R,\\ 0, & r>R.\end{cases}$$

With consideration of (3), applying the integral Hanekl transform $L_H[\vartheta(r, z)] = \theta(p, z)$ to Eq. (2), we find

$$\theta(p, z)=A(p)\exp(-pz). \qquad (5)$$

In order to determine the coefficient A(p) we can use the transformed boundary condition (4):

$$\left[{}_\Delta b+\frac{{}_\Delta\alpha}{\lambda}\vartheta(r_*, 0)\right]\frac{RJ_1(pR)}{p}=\left.\frac{d\theta}{dz}\right|_{z=0}-\frac{\alpha_0}{\lambda}\theta(p, 0),\ 0\le r_*\le R, \qquad (6)$$

in which the quantity $\vartheta(r_*, 0)$ appropriate to the additional definition has been introduced through the theorem on the average value for the calculation of the integral

$$\int_0^\infty rJ_0(pr)\vartheta(r, 0)\sigma(r-R)\,dr=\vartheta(r_*, 0)\frac{RJ_1(pR)}{p}, \qquad (7)$$

J_0 and J_1 are Bessel functions of the first kind, of orders 0 and 1.

Substitution of expression (5) into (6), with consideration of (7), yields the value

$$A(p)=-\frac{R^2}{pR+\zeta_0}\frac{J_1(pR)}{pR}A_*,$$

where

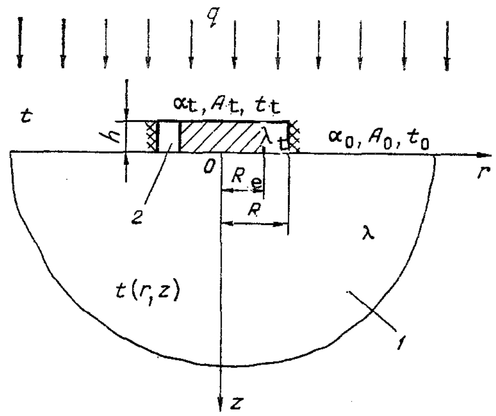

Fig. 1. Heat exchange model: 1) object; 2) heat-metering thermometer.

$$A_* = {}_{\Delta}bR + \zeta_{\Delta}\vartheta(r_*,\ 0);\ \zeta_{\Delta} = \zeta_c - \zeta_0;\ \zeta_c = \frac{1}{1+\zeta_t}\zeta_0\frac{\alpha_t}{\alpha_0}.$$

Applying the inverse Hankel transform to (5), we obtain the sought relationship for the temperature perturbation

$$\vartheta(r,\ z) = -A_*\int_0^\infty \frac{J_1(x)\,J_0(\rho x)\exp(-x\bar{z})\,dx}{x+\zeta_0}, \tag{8}$$

where x is the variable for which the integration is carried out; $\rho = r/R$; $\bar{z} = z/R$.

On the basis of Eq. (8) we determine the change in the thermal perturbation within the object in the radial direction and through the depth:

$$\frac{\vartheta(r,\ 0)}{\vartheta(0,\ 0)} = \frac{\int_0^\infty \frac{J_1(x)\,J_0(\rho x)\,dx}{x+\zeta_0}}{\int_0^\infty \frac{J_1(x)\,dx}{x+\zeta_0}},\quad \frac{\vartheta(0,\ z)}{\vartheta(0,\ 0)} = \frac{\int_0^\infty \frac{J_1(x)\exp(-x\bar{z})\,dx}{x+\zeta_0}}{\int_0^\infty \frac{J_1(x)\,dx}{x+\zeta_0}} \tag{9}$$

with respect to the maximum perturbation in temperature $\vartheta(0,\ 0)$ at the center of the area occupied by the heat-metering thermometer.

As we can see from Fig. 2, with an increase in the Biot number ζ_0 (an increase in the heat-transfer coefficient α_0) we have a reduction in the zone in which the temperature is perturbed within the object, as a consequence of the presence of the thermometer. The dimensions of this perturbation zone in the r direction are considerably smaller than those of the perturbation zone in the z direction: thus, for the case $\zeta_0 = 1$ the magnitude of the ratio $\vartheta(0,\ z)/\vartheta(0,\ 0)$ in the case of z/R = 2, 5, and 10 amounts, respectively, to 0.13, 0.03, and 0.009, while the quantity $\vartheta(r,\ 0)/\vartheta(0,\ 0)$ in the case of r/R = 2, 5, and 10 is, respectively, equal to 0.08, 0.007, and 0.001.

In actual designs the sensing element of a thermometer occupies but a small portion of its volume, bounded by the radius R_e. The thermometer readings correspond to the magnitude of the complete heat flow Q_e, passing through the segment $0 \le r \le R_e$ of area $S = \pi R_e^2$ in which the sensing element of the thermometer is located:

$$Q_e = -\pi R_e^2\lambda\left\langle \frac{dt(r,\ z)}{dz}\bigg|_{z=0}\right\rangle = Q_0 - \pi R_e^2\lambda\left\langle \frac{d\vartheta(r,\ z)}{dz}\bigg|_{z=0}\right\rangle,$$

where $Q_0 = -\lambda\pi R_e^2 b$ is the real value of the total heat flow (in the absence of a thermometer) through that same area: $\left\langle \frac{dt(r,\ z)}{dz}\Big|_{z=0}\right\rangle$ and $\left\langle \frac{d\vartheta(r,\ z)}{dz}\Big|_{z=0}\right\rangle$ represent the values of the temperature and perturbation gradients averaged over the area S.

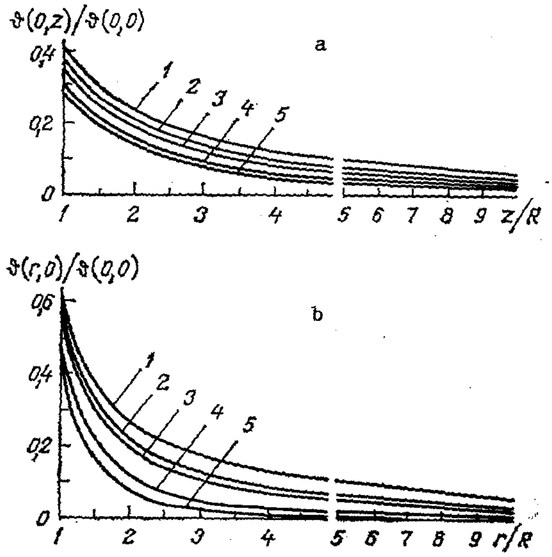

Fig. 2. Temperature perturbation within the depth of the object (a) and in the radial direction (b) relative to the maximum perturbation of the temperature $\vartheta(0, 0)$ in the center of the area occupied by the heat-metering thermometer as a function of the Biot criterion ζ_0: 1) $\zeta_0 = 0$; 2) 0.05; 3) 0.1; 4) 0.5; 5) 1.0.

The relative error in the measurement of the heat flow is determined in the form

$$\delta = \frac{Q_e - Q_0}{Q_0} = \frac{1}{b}\left\langle \frac{d\vartheta(r, z)}{dz}\bigg|_{z=0} \right\rangle. \tag{10}$$

The average value of the temperature perturbation gradient in the segment $0 \le r \le R_e$, found in Eq. (10), can be found by means of the solution for (8):

$$\left\langle \frac{d\vartheta(r, z)}{dz}\bigg|_{z=0} \right\rangle = \frac{2A_*}{R_e}\int\limits_0^\infty \frac{J_1(x)\, J_1(\rho_e x)\, dx}{x + \zeta_0}, \tag{11}$$

in which the value of A_* is determined from the condition that this solution satisfy boundary conditions (4), averaged over the area πR_e^2

$$A_* = \frac{R_e b}{2}\left[\int\limits_0^\infty \frac{J_1(x)\, J_1(\rho_e x)\, dx}{x + \zeta_0} + \zeta_c \int\limits_0^\infty \frac{J_1(x)\, J_1(\rho_e x)\, dx}{x(x + \zeta_0)}\right]^{-1}. \tag{12}$$

Here $\rho_e = R_e/R$.

Substituting (12) into (11), and then (11) into (10), we obtain the sought relationships to estimate the absolute measurements errors $\Delta(\rho_e)$ and the relative measurement errors $\delta(\rho_e)$ for the flow of heat at the surface of the object being investigated:

$$\Delta(\rho_e) = Q_e - Q_0 = \pi R_e^2 q_0 K [1 + \zeta_c \Phi(\zeta_0, \rho_e)]^{-1}, \tag{13}$$

$$\delta(\rho_e) = \frac{Q_e - Q_0}{Q_0} = K [1 + \zeta_c \Phi(\zeta_0, \rho_e)]^{-1}, \tag{14}$$

where

$$K = \frac{1}{1 + \zeta_t}\frac{q_t}{q_0} - 1; \tag{15}$$

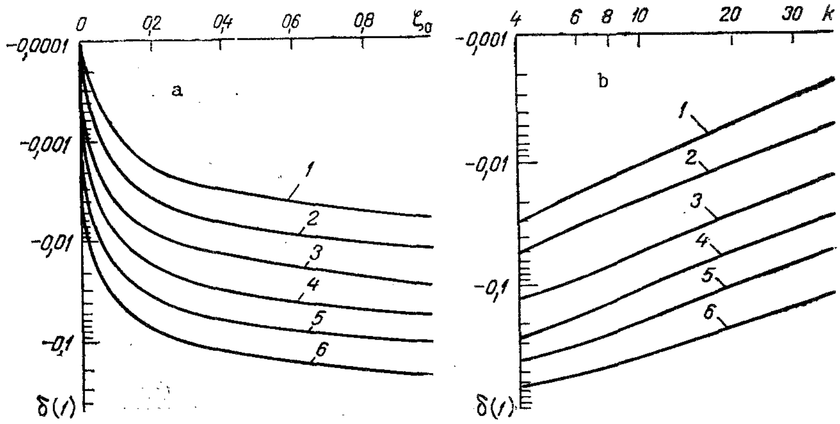

Fig. 3. The errors δ (1) in the thermometer readings as a function of ζ_0 when k = 20 (a) and as a function of the parameter k when ζ_0 = 1 (b) for λ_t/λ = 10 (1), 5 (2), 2 (3), 1 (4); 0.5 (5), 0.2 (6).

$$\Phi(\zeta_0, \rho_e) = \frac{\int_0^\infty \frac{J_1(x)J_1(\rho x)\,dx}{x(x+\zeta_0)}}{\int_0^\infty \frac{J_1(x)J_1(\rho_e x)\,dx}{x+\zeta_0}}; \tag{16}$$

$q_t = \alpha_t(t - t_0) + A_t q$; $q_0 = \alpha_0(t - t_0) + A_0 q$.

Assuming in (16) that $\rho_e = 1$ and $\rho_e = 0$, from (14) we obtain expressions to estimate error for the case in which the sensing element occupies the entire surface of the thermometer ($R_e = R$) or is concentrated at the point:

$$\delta(1) = K[1 + \zeta_c\Phi(\zeta_0, 1)]^{-1}, \tag{17}$$

$$\delta(0) = K[1 + \zeta_c\Phi(\zeta_0, 0)]^{-1}. \tag{18}$$

The numerical values of the integral $\Phi(\zeta_0, \rho_e)$, depending on ζ_0 and ρ_e, have been calculated by means of a computer and can be found in Table 1.

To refine the regions in which expressions (13) and (14) are effective, unlike condition (7) obtained on the basis of the theorem of the average value, we will make use of the method of finding $\vartheta(r_*, 0)$ from the condition of local satisfaction of solution (8) with respect to boundary condition (4), when the parameter r_* assumes the value of the instantaneous coordinate r. Then the expression for the distortion of the temperature at z = 0 can be written in the following form:

$$\vartheta(r, 0) = -A_* I(\zeta_0, \rho), \tag{19}$$

where

$$A_* = \frac{{}_\Delta bR}{1 + \zeta_\Delta I(\zeta_0, \rho)}, \quad I(\zeta_0, 0) = \int_0^\infty \frac{J_1(x)J_0(\rho x)\,dx}{x+\zeta_0}.$$

The temperature perturbation $\vartheta(r, 0)$ at the surface of the object with respect to the maximum perturbation $\vartheta(0, 0)$ is determined by the relationship

$$\frac{\vartheta(r, 0)}{\vartheta(0, 0)} = \frac{I(\zeta_0, \rho)}{I(\zeta_0, 0)}\,\frac{1 + \zeta_\Delta I(\zeta_0, 0)}{1 + \zeta_\Delta I(\zeta_0, \rho)}, \quad I(\zeta_0, 0) = \int_0^\infty \frac{J_1(x)\,dx}{x+\zeta_0}. \tag{20}$$

Let us present the integral $I(\zeta_0, \rho)$ in the form of $I(\zeta_0, \rho) = I(\zeta_0, 0)f(\rho)$ and let us make use of the approximation $f(\rho) = 1 - A\rho^n$. Then instead of (20) we obtain the expression

$$\frac{\vartheta(r, 0)}{\vartheta(0, 0)} = \frac{1}{1 + \zeta_c\Phi(\zeta_0, 0)}\left\{1 + \zeta_c\Phi(\zeta_0, 0) - \frac{1}{1 - \zeta_0 I(\zeta_0, 0)}\right.$$

TABLE 1. Values of the Integral $\Phi(\zeta_0, \rho_e)$

ζ_0	ρ_e										
	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
0	1	0,997	0,993	0,987	0,978	0,966	0,951	0,933	0,910	0,883	0,849
0,01	0,986	0,982	0,978	0,972	0,963	0,951	0,936	0,917	0,894	0,867	0,834
0,02	0,973	0,972	0,968	0,962	0,952	0,940	0,925	0,906	0,883	0,855	0,822
0,05	0,953	0,951	0,947	0,940	0,931	0,918	0,903	0,883	0,860	0,831	0,797
0,1	0,930	0,929	0,925	0,918	0,908	0,895	0,879	0,859	0,834	0,804	0,769
0,5	0,871	0,869	0,864	0,855	0,843	0,827	0,807	0,783	0,752	0,715	0,668
1,0	0,856	0,854	0,848	0,838	0,823	0,804	0,781	0,751	0,715	0,671	0,615

TABLE 2. Values of the Integrals $I_e(\zeta_0, \rho_e)$, $I(\zeta_0, 0)$

ζ_0	$I(\zeta_0, 0)$	ρ_e									
		0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
0	1	0,050	0,099	0,148	0,196	0,242	0,285	0,326	0,364	0,397	0,424
0,01	0,980	0,049	0,097	0,144	0,191	0,235	0,278	0,318	0,355	0,387	0,412
0,02	0,954	0,048	0,095	0,142	0,187	0,231	0,272	0,312	0,347	0,378	0,403
0,05	0,909	0,045	0,090	0,135	0,178	0,220	0,259	0,296	0,330	0,359	0,382
0,1	0,851	0,042	0,085	0,126	0,166	0,205	0,242	0,277	0,308	0,335	0,356
0,5	0,607	0,030	0,060	0,090	0,119	0,146	0,173	0,197	0,219	0,237	0,250
1,0	0,461	0,023	0,046	0,068	0,090	0,111	0,132	0,150	0,167	0,181	0,190

$$\times \left[1 - \frac{I(\zeta_0, \rho)}{I(\zeta_0, 0)}\right]\Bigg\}, \quad (21)$$

where

$$\Phi(\zeta_0, 0) = \frac{I(\zeta_0, 0)}{1 - \zeta_0 I(\zeta_0, 0)}.$$

On the basis of (21) we can find an equation, different from (14), for the measurement error in the stationary heat flow:

$$\delta(\rho_e) = \frac{Q_e - Q_0}{Q_0} = \delta(0)[1 + \varphi(\rho_e)], \quad (22)$$

in which $\delta(0)$ is found from (18), and

$$\varphi(\rho_e) = \frac{\zeta_c \Phi(\zeta_0, 0)}{1 + \zeta_\Delta I(\zeta_0, 0)}[1 + \Psi(\zeta_0, \rho_e)],$$

where

$$\Psi(\zeta_0, \rho_e) = \frac{2}{\rho_e} \frac{I_e(\zeta_0, \rho_e)}{I(\zeta_0, 0)}; \quad I_e(\zeta_0, \rho_e) = \int_0^\infty \frac{J_1(x) J_1(\rho_e x) dx}{x(x + \zeta_0)}.$$

In the particular case, as $\rho_e \to 0$, $\Psi(\zeta_0, \rho_e) = 1$, so that the component $(\rho_e) = 0$.

The numerical values of the integrals $I_e(\zeta_0, \rho_e)$, $I(\zeta_0, 0)$ depending on ζ_0 and ρ_e are shown in Table 2, while the values of $\Phi(\zeta_0, 0)$ can be found in Table 1.

Comparison of the calculation results for the error δ on the basis of formulas (14) and (22) for values of $\rho_e = 0.5$ and 1; $\zeta_0 = 0.01, \ldots$; $\lambda_t/\lambda = 0.5$ and 2 demonstrates their good agreement; the maximum deviation amounts to 1% and is observed when $\rho_e = 1$. This allows us to recommend structurally simpler expressions (13) and (14) for purposes of estimating the static characteristics of the thermometer, situated at the surface of the massive object, for the range of changes in the geometric (ρ_e, k) and thermophysical (λ_t/λ, ζ_0) parameters being investigated here, and namely: $0 \leq \rho_e \leq 1$, $4 \leq k \leq 40$, $0.2 \leq \lambda_t/\lambda \leq 10$, $0 \leq \zeta_0 \leq 1$.

Analysis of relationships (13) and (14) shows that the differences between the heat-exchange factors (α_0, A_0, α_t, A_t) of the object and the sensor, as well as the relationships of the thermal resistance h/λ_t of the thermometer and the thermal resistance $1/\alpha_t$ encountered in the transfer of heat to the ambient medium, to the extent that these affect the magnitude of the measurement error in the flow of heat can be determined by the complex K [see formula (15)]. The function $\Phi(\zeta_0, \rho_e)$, determined from formula (16), characterizes the effect of the exchange of heat in the perturbed region of the object beneath the thermometer, with the error being reduced as the intensity ζ_0 of the exchange of heat is increased.

From (13) follows the condition of error compensation $\Delta(\rho_e) = 0$, if we can satisfy the requirement K = 0, i.e.,

$$\frac{1}{1+\zeta_t} = \frac{q_0}{q_t} = \frac{A_0}{A_t}\,\frac{1+\dfrac{\alpha_0(t-t_0)}{A_0 q}}{1+\dfrac{\alpha_t(t-t_0)}{A_t q}}.$$

In the particular case, when $\alpha_t = \alpha_0$

$$q(A_t - A_0) = \zeta_t[\alpha_0(t-t_0) + A_0 q]$$

and the minimization of the error reduces to the selection of a coating for the thermometer that is close in terms of radiation properties to the properties of the object, i.e., $A_t \approx A_0$.

If the conditions for the exchange of heat at the surface of the object and those of the thermometer are identical ($\alpha_t = \alpha_0$, $A_t = A_0$), i.e., $q_t = q_0$, then, as follows from (15), $K = -(1 + 1/\zeta_t)^{-1}$ and the reduction in the error δ is achieved by reducing the thermal resistance of the thermometer, i.e., by a reduction in the ζ_t number.

In the absence of convective heat exchange ($\alpha_0 = \alpha_t = 0$) the measurement error does not depend on ρ_e and is determined exclusively by the difference in the absorptive capacities of the thermometer and the object:

$$\delta(\rho_e) = \frac{A_t}{A_0} - 1.$$

As an example of utilizing relationships (13) and (14) we undertook a calculation of the error $\delta(1)$ in the measurement of the heat flow, based on formula (17), dependent on the parameter $\zeta_0 = \alpha_0 R/\lambda$ (Fig. 3a) and the geometric parameter $k = 2R/h$ (Fig. 3b) for various values of the ratio of the thermometer thermal conductivity λ_t and the thermal conductivity λ of the object (for the case in which the sensing element of the thermometer occupies its entire surface $R_e = R$ and $A_t = A_0$, $\alpha_t = \alpha_0$, $q_t = q_0$).

Utilization of the above-proposed calculation relationships and graphs allows us to make practical recommendations with regard to optimizing the design of heat-flow measurement converters, wherein consideration is given to the required measurement accuracy, and also to provide for metrological calibrational verification of the sensors, both in the design stage, and under operational conditions.

NOTATION

r, z, axial and radial coordinates; r_*, the value of r for which the mean-valued theorem is satisfied; λ, λ_t, thermal conductivity of the object and the thermometer, respectively, W/(m·K); α_0, α_t, coefficients of convective-radiative heat exchange of the object and the thermometer, respectively, W/(m²·K); A_0, A_t, coefficients of absorption for the object and the thermometer, respectively; R, h, radius and thickness of the thermometer, m; R_e, radius of the sensing element in the thermometer, m; t, temperature of the ambient medium, K; t_0, surface temperature, K: b, temperature gradient, K/m; ϑ, temperature distortion, K; Q, heat flow, W; δ, relative measurement error for the heat flow; Δ, absolute measurement error for the heat flow, W; p, Hankel transform parameter, m^{-1}.

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